J. Fluid Mech. (1969), *vol.* 38, *part* 3, *pp.* 481–498
Printed in Great Britain

Separation and magnetohydrodynamics

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(Received 9 September 1968 and in revised form 29 January 1969)

This paper is an investigation of MHD boundary layers in a transverse magnetic field when the magnetic Reynolds number (*Rm*) is small. The main purpose is to understand something about the suppression of separation by a strong magnetic field, with particular emphasis on the behaviour near a rear stagnation point. Given an $O(1)$ inviscid flow it is shown that there is a critical value of N, the interaction parameter, to completely suppress separation. This value is one half that proposed by Leibovich (1967), a discrepancy that is due to the non-regularity of the boundary-layer equations at a rear stagnation point, a possibility that Leibovich did not consider in his solution. Model linear equations suggest the true role of Leibovich's solution. The possibility of a viscous wake leaving the rear stagnation point is considered and it is suggested that one does *not* arise from vorticity generated in the boundary layer.

Introduction

This work aims to throw additional light on the question of the suppression of separation by a magnetic field. There seems to be no doubt that, granted the exterior inviscid flow, the suppression of vorticity by the non-conservative Lorentz force can delay the onset of separation. Interest then shifts to the neighbourhood of the rear stagnation point in an effort to discover whether the suppression can be complete. This important step was first taken by Leibovich (1967). However, his main conclusion that a sufficiently strong magnetic field does suppress completely is only a part of the rear-stagnation-point problem. Unlike flow at the front stagnation point, the boundary-layer equations at the rear are not regular and attempts to find a Blasius-like expansion as a power series in distance along the wall must fail. Once this is realized the failure of Leibovich's equations for a *weak* field cannot be interpreted as necessarily implying separation. More probably the regularity assumption fails to all orders in distance and a more complex (non-similar) solution must be sought. The critical value of the interaction parameter to prevent separation is one half of the value proposed by Leibovich. For large values of N we suggest that Leibovich's solution is correct if interpreted as the leading term in an *asymptotic* expansion about the rear stagnation point.

Much of our discussion is based on linear model equations, but first of all we examine the proper boundary-layer equations and present evidence that separation can be suppressed. This includes a review of previous work with special attention to that of Leibovich. Model equations and other evidence reveal the non-regularity of the boundary-layer equations at the rear stagnation point, at the same time suggesting the true role of Leibovich's solution. A model for the Navier–Stokes equation throws light onto the nature of the breakdown of the boundary-layer equations implied by the non-regularity. This manifests itself not as a boundary-layer-like wake leaving the rear stagnation point but simply as a neighbourhood of the point of radius $O(R^{-\frac{1}{2}})$ in which derivatives parallel to the wall are as important as those normal to the wall. If the interaction parameter based on the minimum velocity gradient is larger than 1 there should be no observable vorticity shed by the body.

In spite of the positive nature of the above it should be emphasized at an early stage that very little is actually proved. The picture presented is plausible in the light of the few facts that are proved, intuition, and model equations. In addition, the argument is buttressed with a numerical integration of the exact boundary-layer equations.

Flow model

Consider a cylinder with generators perpendicular to the plane of flow, immersed in a conducting fluid and carrying a magnetic field perpendicular to the surface. The strength of the magnetic field is supposed to remain constant around the perimeter. Then assuming that the magnetic field induced by the motion can be neglected, the equations governing the boundary layer are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= U\frac{dU}{dx}+N(U-u)+N\frac{\partial^2 u}{\partial y^2},\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \quad (1)$$

Here U is the outer inviscid flow speed and N is the interaction parameter

$$N = \frac{\sigma B_0^2 L}{\rho U_0}.$$

All quantities are non-dimensional and both y and v have been scaled to eliminate R from the equations. At all times it is assumed that U is an $O(1)$ function of x independent of N. In any application where the outer inviscid flow is known the required modifications should be trivial.

An alternative form of the equations is

$$U\left(\frac{u}{U}\right)\frac{\partial}{\partial x}\left(\frac{u}{U}\right)+v\frac{\partial}{\partial y}\left(\frac{u}{U}\right)+\left(\frac{u}{U}\right)^2\frac{dU}{dx} = \frac{dU}{dx}+N\left(1-\frac{u}{U}\right)+N\frac{\partial^2}{\partial y^2}\left(\frac{u}{U}\right),$$

$$U\frac{\partial}{\partial x}\left(\frac{u}{U}\right)+\left(\frac{u}{U}\right)\frac{dU}{dx}+\frac{dv}{\partial y} = 0,$$

which show quite clearly that at a point where U vanishes the equations are singular. Only a subset of all possible solutions will be regular at such a point and there is no *a priori* guarantee that our boundary conditions will be consistent with solutions from that subset.

Solution for large N

When the interaction parameter is very large we seek a solution to (1) in the form†

$$u = u_0 + (1/N)\,u_1 + \ldots, \quad v = v_0 + (1/N)\,v_1 + \ldots,$$

whence

$$\left.\begin{aligned} &\frac{\partial^2 u_0}{\partial y^2} - u_0 = -U, \\ &\frac{\partial^2 u_1}{\partial y^2} - u_1 = -U\frac{dU}{dx} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}, \\ &\frac{\partial^2 u_2}{\partial y^2} - u_2 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_0}{\partial y}. \end{aligned}\right\} \tag{2}$$

Equations (2) determine the u_i and then the v_i are found from the continuity equation. The leading approximation has the well-known Hartmann-layer structure

$$u_0 = U(1-e^{-y}), \quad v_0 = U'(1-e^{-y}-y),$$

with a first correction

$$u_1 = UU'\,e^{-y}(\tfrac{3}{4}y + \tfrac{1}{4}y^2), \quad v_1 = (UU')'\,(\tfrac{1}{4}y^2\,e^{-y} + \tfrac{5}{4}y\,e^{-y} + \tfrac{5}{4}e^{-y} - \tfrac{5}{4}).$$

Higher terms can be calculated with rapidly increasing effort. We will be content to write down the skin friction at the wall up to terms of order N^{-2}:

$$\frac{\partial u}{\partial y}(x,0) = U + \frac{1}{N}\tfrac{3}{4}UU' - \frac{1}{N^2}(\tfrac{47}{288}U(U')^2 - \tfrac{7}{72}U(UU')') + O(N^{-3}). \tag{3}$$

Provided the prescribed outer flow U is analytic we could, in principle, compute an arbitrary number of terms, implying that separation is completely suppressed at least in the limit $N \to \infty$. The only difficulty that our solution exhibits is at the leading edge of sharp-edged bodies where the initial conditions are not satisfied and the displacement thickness does not vanish (figure 1). In an $O(N^{-1})$ neighbourhood of the nose the inertia terms must be reinstated. This is also necessary at the back in order to continue the boundary-layer development off the body. On the other hand, if the body is blunt, it can be sheathed with our solution everywhere without contradiction (figure 2). The remarkable flow depicted by figure 2 in which no vorticity generated at the body surface ever leaves the thin layer sheathing the body is believed possible not just in the limit $N \to \infty$, but for sufficiently large finite N. Equation (3) can be used to crudely estimate what is meant by 'sufficiently large'. Suppose the exterior velocity is

$$U = 2\sin x,$$

so that

$$\left[\frac{1}{\sin x}\frac{\partial u}{\partial y}(x,0)\right]_{x=\pi} = 2 - \frac{3}{N} - \frac{25}{24}\frac{1}{N^2} + \ldots.$$

If the series was convergent it would vanish for N just equal to the value necessary to suppress separation. The first two terms vanish when $N = \frac{3}{2}$; all three vanish when $N \simeq 2{\cdot}02$. The suggestions that interaction parameters of unit order of magnitude are sufficient to suppress separation will subsequently be confirmed.

† Hunt & Leibovich (1967) have described a development of this kind in their work on two-dimensional ducts.

Unsteady analysis

If the exterior velocity is established impulsively, the initial motion is described by a balance between the viscous term and the time derivative. Such an analysis was first done for OHD by Blasius.† Define

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}, \quad \psi = 2\surd(Nt)f, \quad \eta = \frac{y}{2\surd(Nt)},$$

whence $$4tf_{\eta t} - 2\eta f_{\eta\eta} + 4t(f_\eta f_{\eta x} - f_x f_{\eta\eta}) = 4t(UU' + NU) - 4Ntf_\eta + f_{\eta\eta\eta}. \tag{4}$$

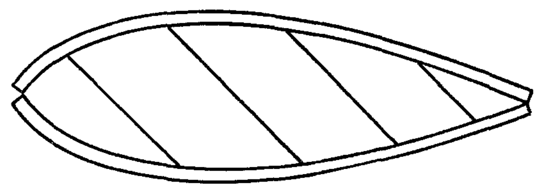

FIGURE 1

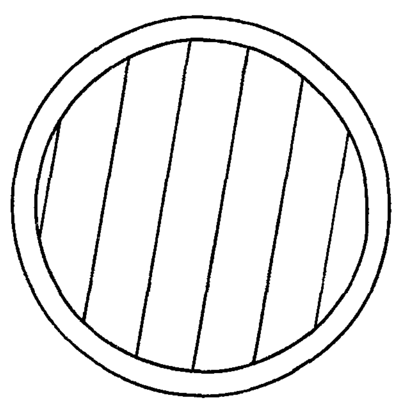

FIGURE 2

A solution to equation (4) is sought in the form

$$f = \sum_{n=0}^{\infty} \zeta_n(\eta, x)\, t^n.$$

The ζ_n satisfy

$$\left.\begin{aligned} \zeta_{0\eta\eta\eta} + 2\eta\zeta_{0\eta\eta} &= 0,\\ \zeta_{1\eta\eta\eta} + 2\eta\zeta_{1\eta\eta} - 4\zeta_{1\eta} &= 4\zeta_{0\eta}\zeta_{0\eta x} - 4\zeta_{0x}\zeta_{0\eta\eta} + 4N\zeta_{0\eta} - 4(UU' + NU),\\ \zeta_{2\eta\eta\eta} + 2\eta\zeta_{2\eta\eta} - 8\zeta_{2\eta} &= 4\zeta_{0\eta}\zeta_{1\eta x} + 4\zeta_{1\eta}\zeta_{0\eta x} - 4\zeta_{0x}\zeta_{1\eta\eta} - 4\zeta_{1x}\zeta_{0\eta\eta} + 4N\zeta_{1\eta}. \end{aligned}\right\} \tag{5}$$

Now write $\zeta_0 = UF_0(\eta)$, $\zeta_1 = UU'F_{10}(\eta) + UF_{11}(\eta)$, so that

$$\left.\begin{aligned} F_0''' + 2\eta F_0'' &= 0,\\ F_{10}''' + 2\eta F_{10}'' - 4F_{10}' &= 4[(F_0')^2 - F_0F_0'' - 1],\\ F_{11}''' + 2\eta F_{11}'' - 4F_{11}' &= 4N(F_0' - 1). \end{aligned}\right\} \tag{6}$$

† *Note added in proof.* It has been pointed out to the author that Tsinober *et al.* (1963) first did an analysis along these lines.

The solution for F_0 is $$F_0' = \operatorname{erf}\eta.$$

Schlichting (1968) gives the solution of F_{10} as derived by Blasius, and the solution for F_{11} is
$$F_{11}' = N\operatorname{erfc}\eta - 4Ni^2\operatorname{erfc}\eta.$$

This is the contribution made by the magnetic field.

The skin friction is given by
$$\frac{f''(0)}{U} = \frac{2}{\sqrt{\pi}} + t\left[U'\frac{2}{\sqrt{\pi}}\left(1+\frac{4}{3\pi}\right)+\frac{2N}{\sqrt{\pi}}\right] + O(t^2). \tag{7}$$

On the back portion of the body where U' is negative it is clear that the magnetic field delays the appearance of the rear vortex. Indeed, if $U = 2\sin x$ then for
$$N = 2(1+\tfrac{4}{3}\pi)$$
the square bracket in (7) vanishes and to that approximation this is what is required to completely suppress separation.

Review of previous work

In 1964 two papers were published that investigated equations (1) with a view to determining the effect on separation of a magnetic field. Fucks, Fischer & Uhlenbusch took U to be proportional to $\sin x$ with a proportionality constant that vanished in the limit $N \to \infty$. They made a Blasius-type expansion about the front stagnation point, numerically integrating the set of similarity equations. Separation was only prevented in the limit $N \to \infty$ but the authors did not claim any accuracy for large N.

Moreau (1964) used Meksyn's method and found the following criterion for separation:
$$\xi = \int_0^x U\,dx, \quad -\xi\frac{d}{d\xi}(\ln U^2) - \frac{2N\xi}{U^2} = 0{\cdot}2. \tag{8}$$

If $U = 2\sin x$ this gives for the location of the separation point
$$\cos x_s = \frac{-(N+0{\cdot}2)}{2{\cdot}2},$$
showing that the critical value of N is 2.

Leibovich (1967) made an important contribution to the problem by concentrating on the rear stagnation point. He showed that the boundary-layer equations derived by assuming a velocity of the form $u = xf'(y)$ only have a solution if $N_L > 2$, where
$$N_L = \frac{N}{-\alpha}, \quad \alpha = U'|_{RSP},$$
where subscript RSP indicates values at the rear stagnation point. Thus for $U = 2\sin x$ he was unable to get a solution for $N \leqslant 4$. Leibovich interpreted this result as indicating that separation and/or unsteady flow occur if $N \leqslant 4$ despite the fact that the skin friction does not vanish when $N = 4$. This curious result contradicts that of Moreau. The value $N = 2$ was only special in Leibovich's

unsteady analysis where it marked the dividing line between unsteady flow with and without eddies.

Another curious feature of the rear stagnation point similarity solution is the algebraic decay for large y. The arguments against algebraic decay break down at isolated points and also if the exterior flow can be algebraically infinite at the body surface at the second approximation in an expansion for large Reynolds number (Brown & Stewartson 1965). In general we do not know the first inviscid approximation to the outer flow so that the second possibility cannot be evaluated, but algebraic decay would be less of a problem if the rear stagnation point was an isolated point in this respect. There is nothing isolated about a point about which there is a uniformly convergent Taylor series in x. Put another way, we would hope that the algebraic decay is a manifestation of the non-commutability of the limits

$$y \to \infty, \quad x \to x_{RSP}.$$

If these two limits do not commute there is no uniformly convergent expansion about x_{RSP}.

Exact separation criterion

Moreau's (1964) result that $N = 2$ is the critical value when $U = 2\sin x$ can be confirmed without solving the equations. It is only necessary to make the same argument that is used to show, in OHD, that separation occurs when the pressure gradient is adverse.

At the separation point the skin friction vanishes but u is strictly positive in some neighbourhood of the wall whence

$$\frac{\partial^2 u}{\partial y^2}(x_s, 0) \geqslant 0. \tag{9}$$

Equations (1) and (9) then imply that, at separation,

$$N + \frac{dU}{dx}(x_s) \leqslant 0. \tag{10}$$

It follows that if N is larger than the maximum value of $(-U')$ on the body (e.g. 2 if $U = 2\sin x$) then the skin friction is positive everywhere on the body and separation does not occur or, more precisely, reversed flow does not occur. The criterion (10) is not altered if the flow is unsteady in agreement with Leibovich's unsteady analysis where for $N > 2$ eddies did not form. When $2 < N < 4$ Leibovich suggested that separation without reversed flow occurred: recall that $N = 4$ was the smallest value for which a rear-stagnation-point similarity solution could be found. Now certainly we do not reject completely the possibility of separation without reversed flow. On the other hand, it will be shown that the failure of a similarity solution does not necessarily imply separation. In addition, numerical solutions of (1) for $U = 1-x$ do not exhibit separation when $1 < N < 2$, although there is only a similarity solution for such a U when $N > 2$. Thus Leibovich's unusual hypothesis, whose physical picture is unclear, is not needed.

Asymptotic solution for large y

It is relevant to our brief discussion of Leibovich's result to consider the general solution of the boundary-layer equations as $y \to \infty$. Following Brown & Stewartson (1965) we suppose

$$\left.\begin{aligned} u &\sim U(x)+A(x,y)\exp\left[-\frac{(y-k(x))^2}{2F(x)}\right],\\ v &\sim -yU'(x)+h'(x), \end{aligned}\right\} \tag{11}$$

where the y dependence of A is algebraic and exponentially smaller terms have been omitted. Substituting (11) into the momentum equation and equating powers of y yield

$$\left.\begin{aligned} \tfrac{1}{2}UF'+U'F &= N,\\ Uk'+U'k &= h',\\ U\frac{\partial A}{\partial x}+y\frac{\partial A}{\partial y}\left(\frac{2N}{F}-U'\right)+A\left(U'+\frac{N}{F}+N\right) &= 0. \end{aligned}\right\} \tag{12}$$

The magnetic field makes two contributions: it introduces an additional term into the equations, namely the last term in the third of (12), and it makes an examination of the solution in the neighbourhood of the rear stagnation point meaningful.

Integrating equations (12),

$$k = \frac{h+k_1}{U}, \quad \frac{F}{N} = \frac{2\int_{x_1}^{x} U\,dx}{U^2}, \tag{13}$$

where k_1 and x_1 are constants. Also we write

$$A = B(x)\,y^n;$$

x_1 and n are determined from the solution of the front stagnation point (the limits $x \to 0$, $y \to \infty$ commute) whence

$$x_1 = 0, \quad n = -(N+3).$$

Integrating the equation for A yields

$$BU^{4+N} = \text{const.}\exp\int dx\left\{\frac{2N+5}{2\int_0^x U\,dx}-\frac{N}{U}\right\}, \tag{14}$$

so that if $U = \sin x$

$$B(\sin x)^{4+N} = \text{const.}\,(\tan\tfrac{1}{2}x)^{-N}\exp\{-\tfrac{1}{2}(2N+5)\cot\tfrac{1}{2}x\},$$

which behaves as $x \to \pi$ like $\quad B \sim (\pi-x)^{-4}.$

Clearly the two limits $x \to \pi$, $y \to \infty$ do not commute.

Attempted regular expansion

The failure of the above limits to commute implies that a regular expansion will not work. It is of interest to see how the failure arises. A solution is sought in the form

$$U = \sum_{i=0}^{\infty} A_{2i+1} x^{2i+1}, \quad u = \sum_{i=0}^{\infty} A_{2i+1} f'_{2i+1}(y)\, x^{2i+1},$$
$$v = -\sum_{i=0}^{\infty} (2i+1) A_{2i+1} f_{2i+1}(y)\, x^{2i},$$

which, when substituted into the boundary-layer equations, yield

$$\left.\begin{aligned} \frac{N}{A_1} f_1'''' - \frac{N}{A_1}(f_1'-1) + 1 + f_1 f_1'' - f_1'^2 &= 0, \\ \frac{N}{A_1} f_3'''' - \frac{N}{A_1}(f_3'-1) + 4 + 3f_3 f_1'' + f_1 f_3'' - 4f_1' f_3' &= 0, \\ \dots \quad \dots \quad \dots\dots \end{aligned}\right\} \tag{15}$$

When y is large
$$f_i \sim y + C_i + F_i$$
and
$$\frac{N}{A_1} F_1'''' - \left(\frac{N}{A_1} + 2\right) F_1' + y F_1'' = 0, \tag{16a}$$
$$\frac{N}{A_1} F_3'''' - \left(\frac{N}{A_1} + 4\right) F_3' + y F_3'' = 4F_1' - 3yF_1''. \tag{16b}$$

For the rear stagnation point it is appropriate to write $N/A_1 = -N_L$. Then with
$$y = Y\surd N_L, \quad F_1' = T\, e^{\frac{1}{4}Y^2}$$
T satisfies the equation $\quad T'' - (\frac{1}{4}Y^2 + (N - \frac{5}{2}))\, T = 0.$

Solutions are the parabolic cylinder functions, whence
$$F_1' \sim Y^{2-N_L},$$
which is only acceptable if $N_L > 2$. This was first discovered by Leibovich.

It is immediately clear that a decaying complementary function for (16*b*) only exists if $N_L > 4$ so that any solution for $N_L < 4$ cannot contain an arbitrary constant. But there are two boundary conditions at the wall, only one of which has been taken care of (by C_3). It follows that a solution for f_3 can, in general, only exist if $N_L > 4$.

This escalation continues to higher terms. At each stage a solution is only possible if
$$N > -(2i+2)\, A_1 \quad (i = 0, 1, 2, \dots), \tag{17}$$
and therefore for all finite N a regular expansion must eventually fail. $N_L = 2$ is no more a special value than $N_L = 4$ or $N_L = 6$. They all represent stages at which an additional regular term can be found, the only distinction of 2 being that it is the *first* value for which any regular term can be calculated. This does not contradict the results of the large-N analysis of course. It simply implies that the two limits $N \to \infty$, $x \to x_{RSP}$ do not commute.

What role then does Leibovich's solution play in a description of rear-stagnation-point flow? It seems plausible that for $N_L > 2$ it is the leading term in an asymptotic expansion about the rear stagnation point. The next term in such an expansion is f_3 *if it can be found* (i.e. $N_L > 4$), but if not one or more non-regular terms must be introduced. If $N_L > 1$ (so that there is no separation) but $N_L < 2$, the leading term in the expansion is non-regular. This idea will be explored using some linear model equations.

Oseen analysis

As our first model equation we take

$$U\frac{\partial u}{\partial x} = U\frac{dU}{dx} + N(U-u) + N\frac{\partial^2 u}{\partial y^2}. \tag{18}$$

Equation (10) is still true for this simplified equation. In addition, an escalation similar to (17) occurs if we attempt a regular solution at the rear stagnation point. However, the failure of the first regular term coincides with the vanishing of the skin friction.

Let $$u - U = e^{-F}P,$$

so that $$\left.\begin{aligned}\frac{\partial P}{\partial F} &= \frac{\partial^2 P}{\partial y^2}.\\ F \text{ is equal to } &\int \frac{N\,dx}{U},\end{aligned}\right\} \tag{19}$$

so that for the special case $U = 2\sin x$ we have

$$F = \tfrac{1}{2}N \ln\tan\tfrac{1}{2}x = N_L \ln\tan\tfrac{1}{2}x.$$

The boundary conditions for P are

$$\frac{P(F,0)}{2} = -e^F \operatorname{sech}\left(\frac{F}{N_L}\right), \quad P(F,\infty) = 0, \tag{20a}$$

and the formulation is complete if the initial conditions are given. It is inconvenient to take the initial station as $x = 0$; instead we take it as $x = \tfrac{1}{2}\pi$, $F = 0$ and

$$P(0,y) = 0. \tag{20b}$$

The solution for P given by Laplace transforms is

$$\frac{P(F,y)}{2} = -\frac{1}{4\pi i N_L}\int_{B_r} ds\, e^{sF-\sqrt{(s)}y}\left[\psi\left(\frac{N_L s}{4}+\frac{3}{4}-\frac{N_L}{4}\right) - \psi\left(\frac{N_L s}{4}+\frac{1}{4}-\frac{N_L}{4}\right)\right],$$

where the logarithmic derivative of the gamma function, or psi function, is

$$\psi(z) = -\gamma + (z-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(z+n)};$$

ψ is mermorphic with simple poles at the non-positive integers.

The velocity is

$$u-U = \frac{4}{N_L^2}e^{-F}\sum_{n=0}^{\infty}\left\{I\left(\frac{4}{N_L}n+\frac{3}{N_L}-1\right)-I\left(\frac{4}{N_L}n+\frac{1}{N_L}-1\right)\right\}, \tag{21}$$

where

$$I(\alpha) = \frac{1}{2\pi i}\int_{B_r} ds\,\frac{e^{sF-\sqrt{(s)}y}}{s+\alpha}.$$

At the rear stagnation point $F \to \infty$ so that it is only necessary to discuss the rightmost singularities in the transform plane. If $N_L < 1$ all the poles lie to the left of the imaginary axis so that the major contribution to u comes from the branch cut terminating at the origin. If $1 < N_L < 3$ just a single pole appears in the right half-plane at $1-1/N_L$. If $3 < N_L < 5$ two poles appear to the right at $1-1/N_L$ and $1-3/N_L$. Each time N_L is increased by a factor of 2 another pole shifts to the right of the origin and every pole in that half of the plane contributes a regular term to the asymptotic expansion. Once all these poles have been used up the branch point contributes an infinite number of terms to the expansion and the first pole in the left half-plane is never reached.

Deforming the contour around the cut leads to the following expressions for I:

$$I = e^{-\alpha F-\sqrt{(-\alpha)}y}+\frac{1}{\pi}\int_0^\infty dr\,\frac{e^{-rF}\sin\sqrt{(r)}\,y}{\alpha-r} \quad (\alpha < 0),$$

$$I = e^{-\alpha F}\cos\sqrt{(\alpha)}\,y+\frac{1}{\pi}(P)\int_0^\infty dr\,\frac{e^{-rF}\sin\sqrt{(r)}\,y}{\alpha-r} \quad (\alpha > 0),$$

and the integral, the contribution from the cut, is written as

$$\frac{1}{\pi\alpha F}\sum_{m=0}^{\infty}\int_0^\infty dt\,e^{-t}\sin\left(\sqrt{t}\frac{y}{\sqrt{F}}\right)\left(\frac{t}{\alpha F}\right)^m.$$

Thus the asymptotic expansion for u when $1 < N_L < 3$ is

$$u-U \sim -\frac{4}{N_L^2}\exp\left[\frac{-F}{N_L}-y\sqrt{\left(1-\frac{1}{N_L}\right)}\right]+\frac{4}{N_L^2}\frac{e^{-F}}{\pi F}\sum_{m=0}^{\infty}\int_0^\infty dt\,e^{-t}\sin\left(\sqrt{t}\frac{y}{\sqrt{F}}\right)\left(\frac{t}{F}\right)^m$$
$$\times\sum_{n=0}^{\infty}\left[\left(\frac{4}{N_L}n+\frac{3}{N_L}-1\right)^{-m-1}-\left(\frac{4}{N_L}n+\frac{5}{N_L}-1\right)^{-m-1}\right]. \tag{22}$$

It is consistent with the order of the expansion to replace $\frac{1}{2}U$ by $s(\equiv \pi-x)$ and F by $(-N_L\ln s)$ in (22). Should the inequality $3 < N_L < 5$ hold, the two leading terms would be $O(s)$ and $O(s^3)$ respectively, followed by the logarithmic terms.

Suppose now we seek the asymptotic expansion directly from (18) by writing

$$\tfrac{1}{2}U = sf(y)+s^{N_L}\sum_{n=1}^{\infty}\frac{g_n(b)}{(\ln s)^n}, \tag{23}$$

where

$$t = y/\sqrt{(-\ln s)}.$$

For $N_L < 3$ we are forced to end the regular terms at s since higher-order terms diverge algebraically as $y \to \infty$. We find

$$f = 1-\exp\left[-\sqrt{(1-1/N_L)}\,y\right]$$

in agreement with the first term of (22), and the functions g_n satisfy

$$N_L g_n'' + \tfrac{1}{2} t g_n' + n g_n = 0,$$
$$g_n(0) = 0, \quad g_1(\infty) = 0,$$

whence
$$g_n = C_n \int_0^\infty dp\, e^{-p} \sin(\sqrt{p}\, t/\sqrt{N_L})\, p^{n-1}, \tag{24}$$

in agreement with the remaining terms of (22). However the C_n are unknown constants that cannot be determined from an examination of the rear stagnation point alone. The solution of a parabolic equation at a point depends on the entire history of the solution up to that point. If this history is not known, it will be reflected in the present kind of indeterminacy. Note however that the regular terms can be found by a purely local study.

A second model equation

A more realistic model equation accounts also for the second inertia term. Consider then

$$-s\frac{\partial T}{\partial s} + y\frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} - NT,$$
$$T(s, \infty) = 0, \tag{25}$$

and T supposed given on the wall. This model can be derived from (1) by writing $T = u - U$, $v = -y$ and $U = -s$. At the wall we should then have

$$T(s, 0) = s, \tag{26}$$

but we shall feel free to generalize (26) to an arbitrary power series in s. Then with

$$T(s, 0) = 1, \quad t = -\ln s \quad \text{(initial station at } t = 0),$$
$$\frac{\sqrt{\pi}}{e^{\frac{1}{4}y^2} 2^{\frac{1}{2}N}} T = \frac{1}{2\pi i}\int_{Br} ds\, e^{st}\, 2^{\frac{1}{2}s} \frac{\Gamma(\frac{1}{2}(N+1+s))}{s} D_{-N-s}(y). \tag{27}$$

The solution given by the calculus of residues is

$$\frac{\sqrt{\pi}}{e^{\frac{1}{4}y^2} 2^{\frac{1}{2}N}} T = \Gamma[\tfrac{1}{2}(N+1)]\, D_{-N}(y) - 2^{-\frac{1}{2}(1+N)} \sum_{n=0}^{\infty} (-1)^n \frac{2^{-n} D_{2n+1}(y)\, e^{-2nt-(1+N)t}}{n!(n+\frac{1}{2}(1+N))} \tag{28a}$$

$$= \Gamma[\tfrac{1}{2}(N+1)] D_{-N}(y) - \tfrac{1}{2} e^{-(N+1)t} 2^{-\frac{1}{2}N} e^{-\frac{1}{4}y^2} \sum_{n=0}^{\infty} \frac{(-1)^n z^n H_{2n+1}(y/\sqrt{2})}{n!(n+\frac{1}{2}(1+N))}, \tag{28b}$$

where the H_n are Hermite polynomials and $z \equiv \frac{1}{4}e^{-2t}$. The expansion (28) is appropriate for small s but is not uniform in y. Fortunately the series can be summed.

Denoting the sum in (28b) by L,

$$z^{1-\frac{1}{2}(1+N)} \frac{d}{dz}(L z^{\frac{1}{2}(1+N)}) = \sqrt{2}\, y \sum_{n=0}^{\infty} z^n 2^{2n} L_n^{\frac{1}{2}}\left(\frac{y^2}{2}\right)$$
$$= \sqrt{2}\, y (1-4z)^{-\frac{3}{2}}\, e^{-2y^2 z/(1-4z)}, \tag{29}$$

where the L_n^α are Laguerre polynomials and the last result is the well-known generating function for these polynomials. In this way

$$\frac{\sqrt{\pi}}{e^{\frac{1}{4}y^2}2^{N/2}}T = \Gamma[\tfrac{1}{2}(N+1)]\,D_{-N}(y) - 2^{\frac{1}{2}(1+N)}e^{-\frac{1}{4}y^2}y\int_0^{\frac{1}{4}e^{-2t}} ds\,\frac{e^{-2y^2s/(1-4s)}}{(1-4s)^{\frac{3}{2}}}\,s^{\frac{1}{2}(1+N)-1}. \tag{30}$$

If $N > 0$ we have the representation

$$D_{-N} = \frac{2^{-\frac{1}{2}(N+1)}}{\Gamma(\frac{1}{2}+\frac{1}{2}N)}\,y\,e^{-\frac{1}{4}y^2}\int_0^\infty e^{-\frac{1}{2}y^2p}\,p^{-\frac{1}{2}(1-N)}(1+p)^{-\frac{1}{2}N},$$

so that after a simple change in variable the final solution is

$$T_M = \frac{2^{\frac{1}{2}(N-M)}}{\sqrt{\pi}}(1-x)^M y\int_{\frac{1}{2}(1-x)^2/(1-(1-x)^2)}^\infty dp\,p^{-\frac{1}{2}}\left(\frac{p}{1+2p}\right)^{\frac{1}{2}(N-M)}e^{-y^2p}, \tag{31}$$

which is actually the solution when the wall condition is

$$T_M(s,0) = s^M.$$

(The substitution $T = sT'$ reduces this problem to the one treated above with $N \to N-M$.) Thus the expansion for small s, (28*a*), is

$$\frac{\sqrt{\pi}}{e^{\frac{1}{4}y^2}2^{\frac{1}{2}(N-M)}s^M}T_M = \Gamma[\tfrac{1}{2}(N-M+1)]D_{-N+M}(y) - 2^{-\frac{1}{2}(1+N-M)}\sum_{n=0}^{\infty}\frac{(-1)^n2^{-n}D_{2n+1}(y)s^{2n+(1+N-M)}}{n!(n+\frac{1}{2}(1+N-M))}. \tag{32}$$

The general solution is a linear combination of such T_M.

When y is large $$D_n(y) \sim e^{-\frac{1}{4}y^2}y^n,$$

so that a regular term $O(s^M)$, decaying for large y, can only be obtained if $N > M$. If $N < M$ the $O(s^M)$ term diverges algebraically as $y \to \infty$ and at the same time fractional powers of s are reached in the expansion. This is somewhat different from the Oseen model. There, breakdown coincided with the appearance of an infinite number of logarithmic terms and the next regular term was never reached. Here the regular terms become interspersed with fractional powers as soon as they are no longer uniformly valid. Note that the leading term in T is either $O(s^M)$ or $O(s^{1+N})$ so that if $M = 1$ the leading term is $O(s)$ for all non-negative N. In addition, as in the Oseen model, the uniformly valid regular terms can be found from a purely local analysis.

A plausible conclusion to be drawn from the results for these linear equations is that, if terms in a regular expansion for the exact boundary-layer equations can be found, then they are the leading terms in an asymptotic expansion about the rear stagnation point. Any information that they give about quantities like skin friction is correct, for example.

Numerical integration

The present view of rear-stagnation-point flow seems more realistic than that of Leibovich. A crucial test, however, is whether the boundary-layer equations (1) have a proper solution when $1 < N_L < 2$ and the most obvious way to establish this is by numerical integration. This was suggested by the referee and fortunately the suggestion coincided with the opportunity to act. Dr P. G. Williams of University College London has a program for integrating the incompressible boundary-layer equations with an arbitrary pressure gradient; this program has proved of great value in studying the effects of suction on separation. He agreed, very kindly, to modify this program to include the Lorentz force $N(U-u)$ in the momentum equation, and we have been able to integrate the equations all the way to the rear stagnation point for sufficiently large N. This tool will enable us to investigate a great many questions about the problem and we hope to report on this in a subsequent paper. For the present purposes we were content to demonstrate two things. In the first place the skin friction at the rear stagnation point is correctly given by the similarity solution when $N_L > 2$. Leibovich gives the skin friction for various values of N and we were able to duplicate these values for $U = 1-x$ by integrating from a uniform flow at $x = 0$. In the second place the flow does not separate when $1 < N_L < 2$, although there is then no similarity solution. Figure 5 is a plot of skin friction at the rear stagnation point as a function of N_L. The values for $N_L \geqslant 2$ are those given by Leibovich and confirmed by us; those for $N_L < 2$ were computed using Williams' program. Our preliminary conclusion is that the skin friction varies perfectly smoothly through the critical value $N_L = 2$ although whether this depends on the form chosen for U remains to be seen.

Reinstatement of the Laplacian

There is no doubt that the boundary-layer equations break down at the rear stagnation point. Expansions of the type (23) are only meaningful at distances large compared with the region of breakdown and at the same time small compared to 1. What is the physical manifestation of this breakdown? In particular, does a thin vortical region leave the rear stagnation point (figure 3) or not? Leibovich, it seems, believed that this did not happen, that instead the boundary layer heals itself (figure 2). He points out that the fluid leaving the boundary layer must lose its vorticity and the Lorentz force is capable of destroying vorticity. This in itself is not enough however. For a flat plate, say, we should certainly not expect the boundary layer to be of the form figure 4 but rather should anticipate a wake as in OHD. The crucial additional fact is the existence of the stagnation point, which ensures that the Lorentz force has a very large time to act. If the boundary-layer thickness is $O(R^{-\frac{1}{2}})$ a fluid particle takes *at least* a time of $O(\ln\sqrt{R})$ to pass through the neighbourhood of the rear stagnation point. The Lorentz force destroys vorticity by a factor e^{-Nt} so that the net decay is $O(R^{-\frac{1}{2}N})$ and if $N > 0$ there is no viscous tail.

This elementary argument can be reinforced by studying the model equation

$$-s\frac{\partial T}{\partial s}+y\frac{\partial T}{\partial y} = \frac{1}{R}\Delta T - NT, \tag{33}$$

which has an obvious relation to (25). The flow situation to be studied is shown in figure 6. T is even in y so that it is only necessary to prescribe the jump in Ty

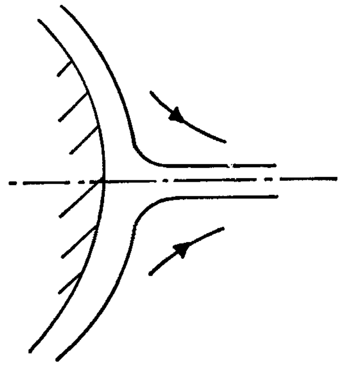

FIGURE 3

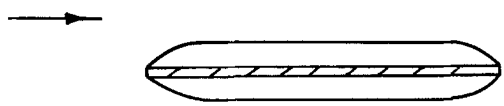

FIGURE 4

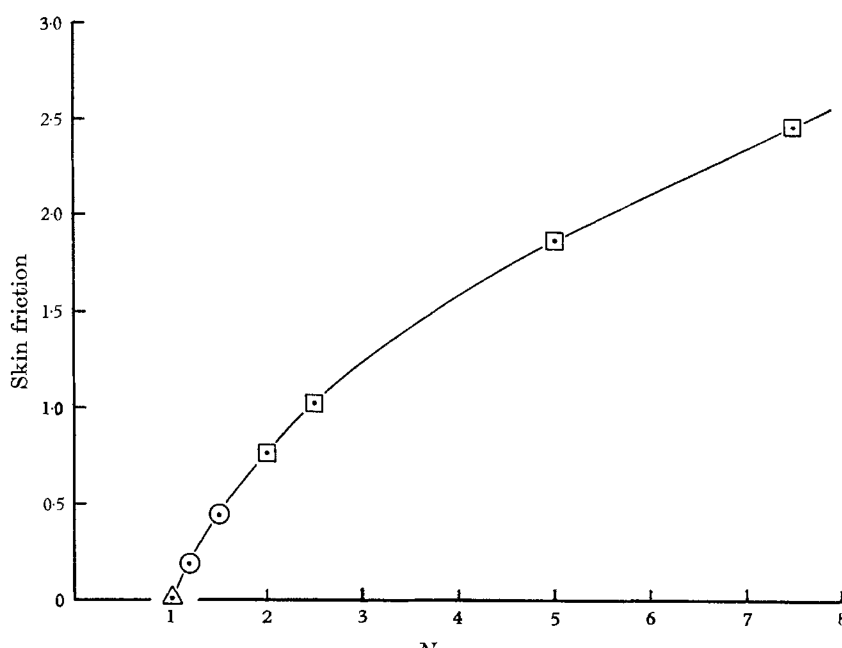

FIGURE 5. ⊡, similarity solution (Leibovich); ⊙, numerical computation; △, separation at rear stagnation point when $N_L = 1$.

across the plate. Boundary layers grow from the edges at ± 1 eventually meeting on the y axis at the rear stagnation point. It is convenient to recast (33) in the form

$$T_{\xi\xi}+T_{\eta\eta}+2\eta T_{\eta}-2\xi T_{\xi}-MT = -\delta(\xi-\alpha)\,\delta(\eta-\beta), \tag{34}$$

because this equation has been investigated by Pillow (1964) in the special case $M = 0$. In the present problem there is a line of sources distributed down the η axis with strength proportional to T_y on the plate. Pillow's derivation of a source solution can be followed for $M \neq 0$ with only minor changes. The substitution $T = e^{-\eta^2}S$ is made and the double Fourier transform of the equation for S is taken. This yields a first-order ordinary differential equation for the transform $\tilde{S}$. Solving for $\tilde{S}$ and then inverting yields

$$T = \frac{\exp\{\frac{1}{2}(\xi^2-\eta^2+\beta^2)\}}{4\pi}\int_{\omega/q}^{\infty}\frac{dt}{t}\left(\frac{t-\omega/q}{t+\omega/q}\right)^{\frac{1}{2}M}\exp\left[-\frac{\omega q}{4}(t+t^{-1})\right],$$

$$\omega \equiv [\xi^2+(\eta-\beta)^2]^{\frac{1}{2}}, \quad q \equiv [\xi^2+(\eta+\beta)^2]^{\frac{1}{2}}, \tag{35}$$

where we have put $\alpha = 0$. The solution (35) makes sense if $M > -2$.

Denoting the solution to (33) by H, and the jump in T_y across the plate by $G(s)$, it follows that

$$H = -\int_{-1}^{+1} dl\, G(l)\, T(s,y;\, l), \tag{36a}$$

$$T(s,y;\,l) = \frac{\exp\{\frac{1}{4}R(y^2-s^2+l^2)\}}{4\pi}\int_{\omega/q}^{\infty}\frac{dt}{t}\exp\left[-\frac{\omega q}{4}(t+t^{-1})\right]\left(\frac{t-\omega/q}{t+\omega/q}\right)^{N}, \tag{36b}$$

$$\omega = \sqrt{\left(\frac{R}{2}\right)}\,[y^2+(s-l)^2]^{\frac{1}{2}}, \quad q = \sqrt{\left(\frac{R}{2}\right)}\,[y^2+(s+l)^2]^{\frac{1}{2}}.$$

A fair amount of information can be deduced from the source solution (36*b*) when the Reynolds number is very large. ωq is $O(R)$ so that the major contribution to the integral in (36*b*) comes from the neighbourhood of $t = 1$ or $t = \omega/q$.

Suppose $\omega/q < 1$ so that

$$|s-l| < |s+l|.$$

This is true for points on the same side of the y axis as the source. Using Laplace's method

$$\sqrt{(R)}T \sim \frac{\sqrt{R}}{4\pi}\sqrt{\left(\frac{4\pi}{\omega q}\right)}\left(\frac{1-\omega/q}{1+\omega/q}\right)^{N}\exp\{\tfrac{1}{4}R(y^2-x^2+l^2)\}\exp\{-\tfrac{1}{4}R[y^4+2y^2(x^2+l^2)+(x^2-l^2)^2]^{\frac{1}{2}}\}. \tag{37}$$

This is an $O(1)$ quantity, the source strength necessary for this being $O(\sqrt{R})$. If $y = 0$, (37) simplifies to

$$\sqrt{R}\,T \sim \frac{\sqrt{R}}{4\pi}\sqrt{\left(\frac{4\pi}{\omega q}\right)}\left(\frac{1-\omega/q}{1+\omega/q}\right)^{N}\exp\{\tfrac{1}{4}R(-s^2+l^2-|s^2-l^2|)\}. \tag{38}$$

If $s > l$ this is exponentially small since the source has very little upstream influence. Downstream of the source ($s < l$) the exponential factor vanishes and the behaviour is algebraic.

Laplace's method can also be used when $s = 0$ so that $\omega/q = 1$. The important contribution then comes from the neighbourhood of the lower integration limit. Then

$$\sqrt{R}\,T \sim \sqrt{R}\,\frac{2^{-N-1}}{4\pi}\left(\frac{4}{\omega q}\right)^{\frac{1}{2}(N+1)}\Gamma[\tfrac{1}{2}(N+1)]. \tag{39}$$

If $N = 0$ this is an $O(1)$ quantity and there is a viscous tail as sketched in figure 3. In general however, this is $O(R^{-\frac{1}{2}N})$ in precise agreement with the argument given at the beginning of the section.

This magnetically produced decay also appears in the vorticity that leaks across the y axis. If $|s+l| > |s-l|$ then $\omega/q > 1$ so that the major contribution to the integral comes from the neighbourhood of ω/q. Then

$$\sqrt{R}\,T \sim \exp\left\{-\frac{R}{2}s^2\right\}\frac{\sqrt{R}}{4\pi 2^N(\omega/q)^{N+1}}\left[\frac{4}{\omega q(1-q^2/\omega^2)}\right]^{N+1}\Gamma(N+1). \tag{40}$$

In addition to the exponential decay of (38) this also has the $R^{-\frac{1}{2}N}$ factor of (39).

Before leaving this section we may point out that a viscous tail can only be generated if T_y on the plate is considerably larger than $O(\sqrt{R})$. Certainly this is the appropriate scale when $s = O(1)$ and it is difficult to imagine that in the Navier–Stokes region close to the rear stagnation point the y derivative can be substantially larger than this.

Navier–Stokes equations

In early discussions of this work a point often raised was that, since Leibovich's solution is an *exact* solution of the Navier–Stokes equations, which presumably have an analytic solution in the neighbourhood of the rear stagnation point, what is the basis then of rejecting it when $N_L < 2$? The answer is presumably that it is not the general solution.

The momentum equation is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = U\frac{dU}{dx}+N(U-u)+\frac{1}{R}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}\right), \tag{41}$$

and we scale the variables thus:

$$v = V/\sqrt{R},\quad y = Y/\sqrt{R},\quad x = X/\sqrt{R},\quad u = u'/\sqrt{R},\quad U = U'/\sqrt{R}.$$

Since $U = -x+a_3x^3+\dots$ we have

$$U' = -x+O(1/R),$$

and seek a solution in the form

$$u = -xf'(y)+x^3g(y)+\dots,$$

whence

$$f'''-ff''+f'^2-1+N(1-f') = 6g. \tag{42}$$

The boundary conditions

$$g(\infty) = 0,\quad f'(\infty) = 0,\quad f(0) = f'(0) = 0,$$

recognize the absence of a viscous tail.

If $g \equiv 0$, (42) is essentially the same as (15) and there are no acceptable solutions if $N < 2$. In general, however, g does not have to vanish identically and there is no reason why difficulties should arise.

The exterior inviscid flow

In the expansion preceding (15) it was assumed that the exterior velocity can be described by a Taylor series about the stagnation point. There is no reason to expect this and it is not an important restriction provided the leading term is

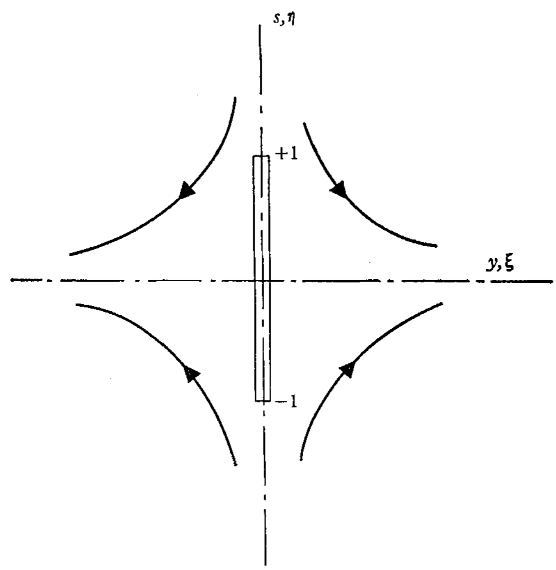

FIGURE 6

$O(x)$. This is important. Presumably $\partial v(0,y)/\partial y$ is neither equal to infinity nor zero so that the continuity equation implies (figure 7)

$$U \sim -x\frac{\partial v}{\partial y}(0,y).$$

Even though the higher-order terms may not be integral powers of x the crucial leading term is $O(x)$, at least when the wall is smooth and the applied field is perpendicular to the wall.

Summary and concluding remarks

In this paper we first showed that suppression of separation can be expected if the magnetic field is strong enough. Crude estimates based on highly restricted analyses indicate that interaction parameters of order 1 are all that are needed for complete suppression. The skin friction was shown to be always positive if $N > \max(-dU/dx)$.

At the rear stagnation point (unlike the front) the two limits $x \to x_{RSP}$, $y \to \infty$ do not commute. It is for this reason that Leibovich's rear-stagnation-point solution has algebraic decay and it implies that a uniform regular expansion about the stagnation point cannot be obtained. The failure reveals itself in practice as a continuously increasing lower limit on N required to compute higher terms. However, we do not reject Leibovich's solution. Linear model equations suggest that if regular terms can be calculated they are correct. Numerical integration of the full boundary-layer equations confirms this.

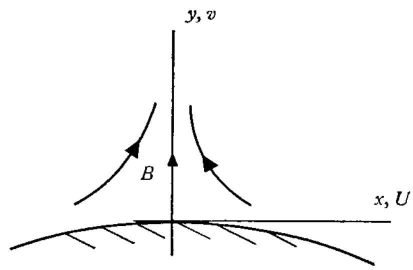

FIGURE 7

In the final sections we examined the nature of the boundary-layer breakdown implied by the non-regularity. Apparently the boundary layer does not leave the body at the rear stagnation point—it heals itself. But, in so doing, derivatives parallel to the wall become as large as those in the normal direction. Any viscous tail can only be associated with non-analyticity in the exterior inviscid flow.

This work was done at Cornell University and forms part of the author's Ph.D. thesis. It was supported there by the U.S. Army Research Office, Durham. Final preparation of the manuscript and the numerical work were done at University College London.

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